The Impact of Constraints on Minimum Variance Portfolios

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Abstract: Minimum variance strategies are a proven approach to profiting from the low volatility effect, but if taken directly from an optimizer they tend to have disadvantageous attributes such as low liquidity, high turnover, high tracking error, and concentrated positions in stocks, economic sectors, and countries or regions. Minimum variance index providers and portfolio managers typically mitigate these implementation problems by imposing constraints. In this study, we construct minimum variance portfolios for the U.S., global developed, and emerging markets, and we apply commonly used constraints to determine their individual and collective impact on simulated portfolio characteristics, investment performance, and implicit trading costs. The constraints we tested succeed in improving investability, but they shift minimum variance portfolio characteristics toward those of the capitalization-weighted benchmark. In particular, each additional constraint increases volatility. Notwithstanding this tendency, the simulated performance advantage of minimum variance indices over market-cap-weighted indices is strong enough to make it a valid choice for investors interested in risk-managed strategies.

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Introduction

In current investment practice, there are two predominant approaches to investing in low volatility stocks. *Optimization strategies* determine portfolio weights through a variance minimization exercise without taking a stand on expected returns. *Heuristic strategies* rank and weight portfolio holdings by a measure of risk, such as total volatility or market beta. Both methods are effective in reducing the volatility of portfolio returns.[1]

In this paper, we will focus on the implementation challenges that accompany optimization-based low volatility investing. Some of these challenges are studied in the literature.[2] Several researchers document poor out-of-sample performance due to estimation errors.[3] Other, more practical issues include extreme and unstable portfolio concentrations, low liquidity, high turnover,[4] and high tracking error.[5] These practical issues prevent the low volatility strategies from being investable. Well-known providers of optimized minimum variance vehicles address such challenges by constraining the portfolio construction process. Contrary to theory, it has been observed that basic restrictions, such as long-only and maximum-weight constraints, improve out-of-sample performance by implicitly reducing estimation errors.[6] Our analysis reveals, however, that investment results may be moderately impaired by other, more stringent constraints that are likewise intended to resolve optimization strategies' implementation issues. We emphasize that these strategies remain strong performers relative to market-capitalization-weighted indices. Nonetheless, investors are well advised to monitor and evaluate results.

We subject to empirical analysis certain constraints similar to those employed by major index providers. By so doing, we hope to offer investors a thorough quantitative and qualitative understanding of the achievable outcome from the investment vehicles available in the marketplace, enabling them to make more informed investment decisions. In addition to examining the constraints' combined impact, we study the effect of each individual constraint separately. Insight into the tradeoff between investability and performance allows investors to choose what is important for them and may induce active quantitative managers to create customized approaches as alternatives to index providers' arguably over-constrained solutions.[7]

Robust in both domestic and international markets, our main finding is that, in the course of making minimum variance portfolios more investable, we move away from the optimal low volatility solution and toward the cap-weighted benchmark. Each additional constraint decreases portfolio turnover or enhances portfolio characteristics at the expense of the low-risk profile. Consequently, minimum variance portfolios which are constrained in the interest of investability have higher volatilities than unconstrained ones. Additionally, in the U.S. and emerging markets (but not in global developed markets),[8] the constraints moderately reduced simulated risk-adjusted returns. Finally, adding constraints is a necessary step in bringing the implicit trading costs of a minimum variance strategy to a tolerable level; even with a full set of constraints, the cost of trading remains significantly higher than it is for the cap-weighted benchmark. Nonetheless, the constraints do not materially erode minimum variance strategies' simulated performance advantage over cap-weighted benchmarks.

We also find that developed market portfolio characteristics, performance, and relative costs are not meaningfully affected by shortening the covariance estimation period or partially rebalancing semiannually rather than fully rebalancing once a year; however, shrinking the universe of eligible stocks from the 1,000 largest to the 500 largest substantially changes the portfolio's liquidity indicators for a modest improvement in absolute performance.

In the next section, we call attention to significant studies that helped us conceptualize our research. We then review our research methodology and define the constraints of interest. Finally, we present our findings, focusing on investability, sectoral and regional concentrations, investment performance, and relative transaction costs.

Literature Review

Minimum variance portfolios are among the several ways in which low volatility strategies have been implemented. Seeking intuitions about the strategies, Clarke, Silva, and Thorley (2011 and 2013) build analytic solutions for long-only minimum variance and other risk-controlled strategies, under the assumption of a single-factor model for the stock returns covariance matrix. These solutions reveal how systematic and idiosyncratic risk affect the relative magnitude of stock weights in portfolios constructed in accordance with the three different methodologies. Clarke, Silva, and Thorley (2006) argue that the minimum variance portfolio is at the left end of the efficient frontier. Behr, Guettler, and Miebs (2008) show that constrained minimum variance portfolios outperform cap-weighted benchmarks; however, they are highly sensitive to revision (i.e., rebalance) frequency and maximum weight constraints. The AGIC researchers (2012) demonstrate that the minimum variance strategy becomes inefficient and falls short of maximizing the Sharpe ratio when constraints are either too loose or too restrictive.

Our practitioner-oriented empirical study contributes to the literature by quantifying the extent to which ad hoc constraints succeed in making minimum variance strategies executable. For instance, they effectively limit turnover to approximately 20% and dramatically lower implicit trading costs from 200–300 times those of cap-weighted benchmarks to 14–21 times.[9] However, the constraints that are thus required for implementing the strategy at scale attenuate its diversifying effect and compromise its power to reduce the overall risk of an investment program.

Empirical Procedures

In this section we describe the data sources, portfolio construction methods, and optimization technique we employed, and we define the constraints whose effects we seek to understand.

Data

We accessed several databases for the historical stock return information used in our tests. We obtained U.S. common stock return information from CRSP, excluding all firm/month observations that lacked contemporaneous return information. For international markets, we took monthly U.S. dollar-denominated stock return data from Datastream. All returns were expressed in U.S. dollars. For robustness, we performed our simulations and tests separately for three markets: the United States, developed markets (including the United States), and emerging markets.[10]

Stock Portfolios

To ensure that portfolio holdings are investable, the starting universe in January of each year contains the 1,000 stocks with the largest market capitalizations as of the prior year-end. Stocks in all markets are eligible for the starting universe only if sufficient monthly return data up to the time of the annual reconstitution are available in advance.

For each market, we build a long-only minimum variance portfolio with an optimization routine under various constraints at the beginning of each January and hold it for one year. We thus obtain simulated return series for several minimum variance equity strategies. For analytical purposes, we also build cap-weighted portfolios from the same starting universes for each market. Each January, these portfolios are reconstituted on the basis of the prior year-end market capitalization data, such that they are synchronized counterparts to the minimum variance portfolios. Constructing the parallel series of cap-weighted portfolios will help us evaluate how the constraints of interest affect the structure of the minimum variance portfolios.

Optimization

At the beginning of each year, up to five prior years of monthly returns are used for convariance matrix estimation and portfolio optimization. Stocks with fewer than three years of immediately past data are excluded from considerations in all markets.

Estimation Errors

Numerous methods are available to overcome the estimation errors inherent in sample covariance matrices.[11] Nonetheless, it has been shown that the various methods generally lead to very similar long-term risk–return characteristics.[12] Commonly used methods include Bayesian shrinkage,[13] principle component analysis (PCA),[14] or sample covariance estimation using more frequently sampled (e.g. daily) series.[15] We select the PCA method to conduct all of our analysis because it estimates covariances directly on the basis of historical data and does not require making assumptions about data structures.[16] In addition, it ensures that the covariance matrix is positive definite, a requirement given that the portfolio optimization takes the inverse of this matrix.[17] The PCA method also helps clearly illustrate the impact of constraints, as others have shown that restricting weights to a predetermined maximum is similar to shrinking the covariance matrix.[18]

Constraints

For the benefit of investors who are considering entering the minimum-variance space, the constraints we impose on the hypothetical minimum variance portfolios closely resemble those applied by leading index providers. They are not only intuitively appealing but also necessary to render these portfolios investable. The constraints are defined as follows:

1. **Minimum weight constraint.** Weights smaller than 0.05% are forced to zero.
2. **Maximum weight constraint.** Individual stock weights are capped at 5%.
3. **Capacity constraint.** The weight of a stock is capped at the lower of 1.5% or 20 times its weight in the corresponding cap-weighted portfolio. Note that this constraint dominates the maximum weight constraint.
4. **Sector concentration constraint.** Sector[19] weights are not allowed to deviate more than ±5% from the corresponding cap-weighted sector weights.
5. **Regional concentration constraint.** If the cap-weighted region weights are less than 2.5%, the minimum variance region weights are capped at three times their weight in the cap-weighted portfolio. Otherwise, they are not allowed to deviate more than ±5% from the corresponding cap-weighted region weights.
6. **Turnover constraint.** The maximum allowable one-way index turnover is 20%.

The economic rationale for the constraints is to make the portfolio more investable. Progressively imposing the constraints naturally moves the minimum variance portfolio allocations toward the allocations observed in the corresponding cap-weighted portfolios.

We impose the constraints stepwise. Only the minimum and maximum weight constraints apply to the base strategy. Then, in all markets, the capacity constraint takes the place of the maximum weight constraint. Next, the regional concentration constraint is added in international markets. After that, the sector concentration constraint is implemented in all markets. Finally, the optimization process is additionally restricted by the turnover constraint, which makes the annual rebalancing path-dependent. (Recall that the year-end weights affect the next year's starting portfolio.) If the optimizer fails to find a perfect solution, then the constraints are made less restrictive, leading to a number of rebalances with one-way turnover above 20%.[20] All of these constraints, except the regional concentration constraint applied to a United States strategy, are binding. In all years, we observe a different portfolio constitution when an additional constraint is imposed.

Additionally, for the purpose of understanding the impact of each constraint, we impose constraints 3−6 individually on the base strategy.

Study Results

Our major findings concern the impact of the selected constraints on the investability, sectoral and regional diversity, and performance and *ex post* risk of minimum variance strategies.

Indicators of Investability

In order to determine whether the constraints under consideration succeed in making minimum variance portfolios more investable, we calculated portfolio characteristics related to liquidity and transaction costs, as well as a proxy of trading cost derived from these characteristics by Aked and Moroz (2015). There are various approaches in the literature to modeling trading costs with the market impact of large trades;[21] we employ Aked and Moroz's model as it decomposes costs due to portfolio characteristics that are intuitive for understanding the capacity of a strategy—turnover, weighted average market capitalization (WAMC), and effective number of holdings (effective *N*).

As shown in **Table 1**, the constraints generally lower turnover. The base strategy requires turning over almost half of the portfolio at each index rebalancing; average turnover is 50% in the United States and global developed markets, and 44% in emerging markets. The explicit turnover constraint very strongly cut these turnover rates down to 20%. Although constraining turnover prevents the optimization from reaching an optimal solution, it very effectively reduces the volume of trades, and it should be considered as investors weigh their options.

The WAMC ratio is the historical average of a portfolio's WAMC as a percentage of the benchmark's WAMC. This is a measure of the average size of portfolio holdings, which is an indicator of the portfolio's investment capacity. As Chan and Lakonishok (1993) and Keim and Madhavan (1997) show, trading small-cap stocks is more costly than trading large-cap stocks. Normalization to the benchmark's WAMC can be viewed as accounting for the fact that market value (as well as trading cost in dollars) is non-stationary. In all markets, the WAMC ratios increase almost monotonically as the constraints take effect. Thus, as intended, the constraints increase the investability of minimum variance strategies. The WAMC ratios of the base and fully constrained portfolios are, respectively, 21% and 45% of the market portfolio in the United States, 27% and 43% in global developed markets, and 18% and 26% in emerging markets.

Effective *N* measures the concentration of the portfolio;[22] a low number characterizes a highly concentrated portfolio, which is difficult to trade. For an understanding of liquidity, effective *N* complements WAMC: a concentrated bet on just a few mega-cap stocks has a very high WAMC, but the low effective *N* of this portfolio reins in its expected investability. A minimum variance portfolio can be very concentrated relative to the cap-weighted benchmark. The average effective *N*s of minimum variance portfolios with very few constraints are only 34, 42, and 33 in the United States, global developed markets, and emerging markets, respectively.[23] Consistently across all markets, the capacity constraint powerfully raises the average effective *N* to the 90−100 range. With all constraints in place, the effective *N* is further increased to above 100. The effective *N* of broad market cap-weighted benchmarks is 150 in the United States and 200 to 300 or higher internationally.[24]

The exact amount of deterioration in performance due to trading costs depends on variables that are specific to prevailing market conditions and individual investors, such as the overall market liquidity at the time of trades and the total assets under management. For the purpose of comparison, we focus our attention on the strategies' cost proxy relative to the cost of trading the cap-weighted benchmark. The cost proxy is defined as

$$Implicit\ Cost\ Proxy = E_t\left[\frac{Turnover}{WAMC\ Ratio\ \cdot Effective\ N}\right]$$

For a fixed amount of assets under management, the implicit cost of trading an indexing strategy is directly proportional to the turnover rate and inversely proportional to the weighted average market capitalization (WAMC) and the number of holdings (effective *N*).[25] The cost of trading the base strategy is 308, 247 and 199 times higher the cost of trading the cap-weighted portfolio for the United States, developed markets and emerging markets, respectively. For example, suppose an investor in a U.S. cap-weighted strategy were to execute transactions that mirrored the rebalancing trades required in a minimum-variance strategy. If she incurred 1 bps of trading costs, an investor engaged in rebalancing the base minimum-variance strategy would suffer an estimated setback of 3.08%.

This high cost makes the unconstrained minimum-variance strategy unattractive.[26] The constraints progressively lower the trading cost. The capacity constraint and the turnover constraint very effectively lower the cost by 3 to 8 times, as the former limits portfolio concentration and overexposure to small-cap holdings, and the latter restricts the amount of trading. The fully constrained strategy has costs that are still significantly higher than the cap-weighted benchmark, but the cost multiple has dropped from the 200-to-300 range down to 14, 21, and 18 for the three regions.

Sectoral and Regional Allocations
A comparative study of the base and fully constrained portfolios' sectoral and regional allocations provides evidence that the constraints of interest push characteristics of minimum variance portfolios in the direction of the corresponding cap-weighted benchmarks. We report the sector allocation trends for the U.S. market only, but we find broadly consistent patterns in the global developed and emerging markets portfolios. In **Figure 1a**, the base minimum variance portfolio leans heavily toward the more stable utilities sector and stays almost entirely out of the more volatile business equipment sector, which includes many technology companies. **Figure 1b** and **Figure 1c** show that the fully constrained portfolio overweights utilities and underweights business equipment, but, due to the explicit sector concentration constraint, the sectoral allocations generally resemble those of the cap-weighted benchmark.

Simulated regional allocations are displayed in **Figures 2a-3c**. The base developed markets portfolio aggressively underweights Japan after the bursting of the Japanese asset price bubble, as it does the United States after the bursting of the dot-com bubble. The base emerging markets portfolio invests more than two-thirds of its value in the "non-BRIC" countries in the EMEA, Asia Pacific, and Americas regions, which include the smallest, least integrated economies. Once again, the constraints significantly shift the minimum variance portfolios' allocations toward those of the corresponding cap-weighted benchmarks.

Performance and Risk Attribution
Our research into the impact of constraints on minimum variance portfolios included computing basic performance statistics for the five different investment strategies[27] in three markets. As **Table 2** shows, the constraints marginally reduced simulated gross performance in the United States, where the return of the fully constrained portfolio was 40 bps lower than the base portfolio return. The constraints improved the absolute returns of the developed and emerging markets portfolios by 100 bps and 50 bps, respectively.

The constraints had a mixed effect on the minimum variance portfolios' risk–return profiles. As Table 2 also reveals, strong monotonic trends in volatilities and tracking errors came to light. With the progressive addition of portfolio constraints, tracking errors vis-à-vis the cap-weighted benchmarks decreased in all three markets. This reveals that additional constraints press the performance as well as the characteristics of minimum variance portfolios toward the corresponding market portfolios. At the same time, we observe a monotonic increase in volatility in all markets. For example, the volatility of the fully constrained emerging markets portfolio is 16.2%, appreciably greater than the 12.1% volatility of the base portfolio. Predictably, adding constraints tends to impair the efficiency of the optimization process in reducing risk.

Reducing volatility less efficiently tends to result in a less favorable Sharpe ratio. At the same time, decreasing the active bets versus the market-capitalization-weighted benchmark improves the information ratio. (These tendencies can also be seen in Table 2.) This trade-off in constrained minimum variance portfolios may be unappealing to investors who measure their returns against total risk rather than benchmark risk.[28]

We also investigated the impact of constraints on the strategies' market betas and sensitivities to other risk factors. We utilized the standard four-factor model—market, size (SMB), value (HML), and momentum (WML),[29] augmented with Frazzini and Pedersen's (2014) "betting against beta" (BAB) factor[30] to capture the low-beta premium. The risk attribution analysis is presented in **Table 3**. We see a monotonic increase in market beta across all markets as we add further constraints to the minimum variance portfolios. This shows that the portfolios gradually take on more systematic risk and, in the process, become more and more correlated with the market.

Note that the fully constrained minimum variance portfolio tends to have lower sensitivity to alternative risk factors than its counterparts with fewer constraints. In the United States, sensitivity to SMB declines from 0.13 to 0.04, and sensitivity to HML drops from 0.17 to 0.06; across all markets, sensitivity to BAB decays with additional constraints.[31]

For the base strategies, which represent a concentrated corner of the markets, the factor model does not adequately explain the risk-return trade off, as indicated by the relatively low R^2 values (78%, 62%, and 61%, respectively, for the United States, the developed markets, and the emerging markets). With all constraints applied, the R^2 becomes very high, around 90% across all markets. These results are also consistent with the observation that the constraints push the minimum variance portfolios toward the market portfolio. All the same, the fully constrained minimum variance portfolio appears to be a sensible alternative to the cap-weighted benchmark: it offers markedly higher Sharpe ratios in all three markets.

Impact of Individual Constraints

In order to demonstrate the stand-alone impact of the constraints, we applied each one separately to the base strategy. The simulated performance and liquidity measures are reported in **Table 4**. The capacity and turnover constraints are both crucial for improving liquidity. The capacity constraint, which limits individual holdings to no more than 1.5% of the total portfolio, meaningfully raises the effective N (i.e., reducing concentration) from 34 to 89 in the United States, 42 to 92 in the developed markets, and 33 to 97 in the emerging markets. As a result, the trading cost is lowered by eight times in the United States and four times in both of the international markets. The improved liquidity comes at a cost, as it significantly increases the portfolio's volatility, especially in the emerging markets.

The turnover constraint is very effective in reducing turnover and trading cost; it consistently lowers the turnover in all markets from almost 50% to just above the 20% target. Compared to the capacity constraint, the turnover constraint has a modest impact on portfolio volatility. The volatility of the emerging markets portfolio increases only from 12.1% to 12.5%. As a practical matter, this observation tends to support imposing a strong turnover constraint and expediently relaxing the parameters if the optimizer fails to find a solution. Under this approach, however, the order and magnitude of the steps taken to moderate the turnover constraint is based upon trial and error, not theory. Index providers who opt for this solution should periodically retest the adjustment routine to ensure it remains effective.

Table 5 and **Table 6** show the consistency of the trade-offs between liquidity and volatility that the capacity and turnover constraints exhibit at various strengths. In the interest of space, we report only the developed market results; the same trend in the liquidity/volatility trade-off, however, can be observed in all three markets.

The sectoral and regional constraints do not improve turnover or holdings-level concentration. Although they do not obviously expand capacity, they are very sensible for investors who are concerned with sectoral and regional concentration risks, because the volatility of portfolios with these constraints remains very attractive relative to the cap-weighted benchmark.

Robustness of the Minimum Variance Methodology

We conducted a series of tests designed to evaluate the robustness of minimum variance strategies in the presence of constraints. They included shortening the historical period on which the covariance matrix is based; reducing the number of eligible stocks; and rebalancing more often than once a year. In this section, we summarize the results for developed market portfolios. All the robustness tests produced similar results in the U.S. and emerging market portfolios. The results are available upon request.

The significant overlapping of sub-periods displayed in Table 1 and Table 2 ensures that the optimizer produces stable portfolios over time but raises a concern that the older data may not reflect current market conditions. As a robustness check, we repeated the analysis with a shorter estimation window comprising up to 36 months of trailing returns (minimally including the most recent 24 months). The performance and liquidity measures for the developed market portfolios are presented in **Table 7a**.

Comparing Table 7a and Table 1, we can clearly observe that using the shorter estimation window leads to higher turnover. When all constraints, including the turnover constraint, are applied, the performance and liquidity measures in Table 7a and Table 1 become very similar, and the volatility, turnover, WAMC and effective *N* remain almost identical. Using a shorter estimation window leads, of course, to a different covariance matrix. Although the covariance matrix is theoretically the only required input for a minimum variance strategy, the presence of multiple (and arguably excessive) constraints attenuates its impact on portfolio characteristics.

The foregoing analyses assume an opportunity set containing the 1,000 largest stocks in a universe. Given that liquidity is central to our study, we conducted a further robustness check by limiting the selection pool to the 500 largest stocks. The effects on developed market portfolio characteristics are reported in **Table 7b**. Shrinking the opportunity set in this manner seems to improve liquidity but modestly diminishes the efficacy of the capacity constraint. Comparing Table 7b and Table 1, investing in the largest 500 stocks augments the WAMC ratio by 15% to 20%. With the selection pool halved, the capacity constraint improves the average effective *N* from 33 to 73; for reference, it raised the average effective *N* from 42 to 92 when the selection pool included the largest 1,000 stocks.

The risk and return profiles of strategies constructed from the 1,000- and 500-stock selection pools are very similar, as shown in Table 7b and Table 2. Nonetheless, when the selection universe is reduced, a few straightforward constraints can prevent the optimizer from computing a feasible solution. In our developed markets simulation, we had to relax the turnover constraint from 20% to more than 30% in order for the optimizer to consistently yield valid solutions.[32] Weakening the turnover constraint tends to offset the gain in liquidity represented by the higher WAMC.

Finally, we studied the impact of different rebalancing frequencies on minimum variance strategies. Our principal results arise from strategies that are rebalanced at the beginning of each calendar year, but leading minimum variance index providers rebalance their strategies semiannually.[33] To isolate the impact of rebalancing frequency, we repeated the analysis with strategies rebalanced at beginning of the first and the third quarters of each year, setting the turnover constraint to 10% rather than 20% in order to hit the same target on an annual basis. The resulting performance and characteristics are reported in **Table 7c**. Comparing these results with those in Table 2, which reflected annual rebalancing, we see that rebalancing frequency has no impact on WAMC and the concentration measures. When the turnover constraint is in force, long-term average turnover stays just above 20%. We also observe that rebalancing more frequently than once a year does not improve performance. Minimum variance strategies do not require frequent rebalancing.

Conclusion

The simulated minimum variance portfolios we tested delivered superior risk-adjusted returns relative to traditional passive investing: In all markets, they produced substantially higher Sharpe ratios than did the cap-weighted benchmarks. Nonetheless, the optimized strategies are difficult to implement efficiently due to their tilt towards smaller companies, their relatively high turnover rates, and their concentrations in stocks, sectors, and countries. These characteristics contribute to low investment capacity and high transaction costs. Minimum variance index providers and managers typically do a good job in controlling trading costs, as well as in improving the strategies' investability, by imposing sensible ad hoc constraints at the security and portfolio levels. However, constraining the portfolio construction process entails greater-than-minimal volatility and tends to shift minimum variance portfolios toward their cap-weighted benchmarks. These findings are fairly consistent across international markets. Even though the performance of the constrained minimum variance portfolios remains markedly superior to that of the cap-weighted benchmarks, investors should be aware of these trade-offs when deciding how to implement a low-volatility strategy.

Appendix: Markets and Regions

We define regions by (1) identifying individual countries which have significant economic scale; (2) grouping small countries together to form significant economic scale, based on their geographical location and commonality in economic drivers.

Global Markets

Region 1 = DevEME, which includes Austria, Belgium, Denmark, Finland, Greece, Ireland, Israel, Italy, Luxembourg, Netherlands, Norway, Portugal, Spain, Sweden, and Switzerland

Region 2 = DevAPAC, which includes Australia, Hong Kong, New Zealand, and Singapore

Region 3 = France

Region 4 = Germany

Region 5 = United Kingdom

Region 6 = Japan

Region 7 = Canada

Region 8 = United States

Emerging Markets

Region 1 = EMEMEA, which includes Czech Republic, Egypt, Hungary, Morocco, Poland, and Turkey

Region 2 = EMAPAC, which includes Indonesia, Malaysia, Philippines, and Thailand

Region 3 = EMAME, which includes Chile, Colombia, Mexico, and Peru

Region 4 = South Africa

Region 5 = Russian Federation

Region 6 = India

Region 7 = China

Region 8 = Taiwan

Region 9 = South Korea

Region 10 = Brazil

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Table 1. Liquidity Indicators

Panel A: United States (Jan. 1967 - Sept. 2014)	Turnover	WAMC Ratio	Effective *N*	Weight in Top 10 Holdings	Implicit Cost Proxy *	Cost Proxy Relative to Benchmark
Simulated Cap-Weighted Benchmark	4.7%	100.0%	150	19.4%	0.33	1.00
Base	49.5%	20.5%	34	44.7%	102.10	307.63
Add Capacity Constraint	36.7%	33.1%	89	15.0%	12.95	39.02
Add Sector Concentration Constraint	38.2%	43.8%	89	15.0%	10.38	31.27
Add Turnover Constraint	20.0%	45.2%	105	15.0%	4.48	13.50
Panel B: Developed Markets (Jan. 1987 - Sept.2014)	**Turnover**	**WAMC Ratio**	**Effective *N***	**Weight in Top 10 Holdings**	**Implicit Cost Proxy ***	**Cost Proxy Relative to Benchmark**
Simulated Cap-Weighted Benchmark	6.5%	100.0%	329	10.6%	0.21	1.00
Base	49.7%	27.1%	42	39.6%	52.87	246.62
Add Capacity Constraint	40.4%	38.0%	92	15.0%	12.80	59.73
Add Region Concentration Constraint	42.7%	39.5%	93	15.0%	13.50	62.97
Add Sector Concentration Constraint	45.2%	42.8%	93	15.0%	12.75	59.49
Add Turnover Constraint	20.2%	43.0%	111	14.8%	4.55	21.22
Panel C: Emerging Markets (Jan. 2002 - Sept. 2014)	**Turnover**	**WAMC Ratio**	**Effective *N***	**Weight in Top 10 Holdings**	**Implicit Cost Proxy ***	**Cost Proxy Relative to Benchmark**
Simulated Cap-Weighted Benchmark	8.4%	100.0%	218	14.7%	0.41	1.00
Base	43.6%	17.6%	33	45.7%	82.47	199.24
Add Capacity Constraint	36.5%	21.1%	97	15.0%	19.00	45.90
Add Region Concentration Constraint	39.1%	24.8%	96	15.0%	17.67	42.70
Add Sector Concentration Constraint	41.0%	26.5%	98	15.0%	17.16	41.47
Add Turnover Constraint	20.2%	26.4%	109	15.0%	7.37	17.80

* Implicit Cost Proxy calculated as historical average of [turnover / (WAMC relative to benchmark x Effect N)] x 1000

Source: Research Affiliates using data from Compustat, CRSP, Worldscope, and Datastream

Table 2. Performance and Risk

Panel A: United States (Jan. 1967 - Sept. 2014)	Return	Volatility	Sharpe Ratio	Return in Excess of Benchmark	Tracking Error	Information Ratio
Simulated Cap-Weighted Benchmark	10.30%	15.40%	0.34	0.00%	0.00%	
Base	12.00%	12.10%	0.57	1.60%	9.20%	0.18
Add Capacity Constraint	11.20%	12.30%	0.5	0.90%	7.50%	0.12
Add Sector Concentration Constraint	11.70%	12.90%	0.51	1.40%	6.00%	0.23
Add Turnover Constraint	11.60%	13.00%	0.5	1.30%	5.50%	0.23
Panel B: Developed Markets (Jan. 1987 - Sept. 2014)	**Return**	**Volatility**	**Sharpe Ratio**	**Return in Excess of Benchmark**	**Tracking Error**	**Information Ratio**
Simulated Cap-Weighted Benchmark	7.70%	15.60%	0.27	0.00%	0.00%	
Base	7.40%	10.30%	0.38	-0.30%	10.70%	-0.03
Add Capacity Constraint	8.50%	10.90%	0.46	0.70%	9.30%	0.08
Add Region Concentration Constraint	8.20%	11.50%	0.41	0.50%	8.50%	0.06
Add Sector Concentration Constraint	8.20%	12.10%	0.39	0.50%	7.20%	0.06
Add Turnover Constraint	8.40%	12.40%	0.4	0.70%	6.40%	0.11
Panel C: Emerging Markets (Jan. 2002 - Sept. 2014)	**Return**	**Volatility**	**Sharpe Ratio**	**Return in Excess of Benchmark**	**Tracking Error**	**Information Ratio**
Simulated Cap-Weighted Benchmark	13.20%	22.20%	0.53	0.00%	0.00%	
Base	16.40%	12.10%	1.24	3.20%	15.00%	0.21
Add Capacity Constraint	19.10%	14.50%	1.22	5.90%	11.70%	0.51
Add Region Concentration Constraint	17.60%	15.20%	1.06	4.40%	10.00%	0.44
Add Sector Concentration Constraint	16.40%	15.60%	0.96	3.20%	9.50%	0.34
Add Turnover Constraint	16.90%	16.20%	0.95	3.80%	8.60%	0.44

Source: Research Affiliates using data from Compustat, CRSP, Worldscope, and Datastream

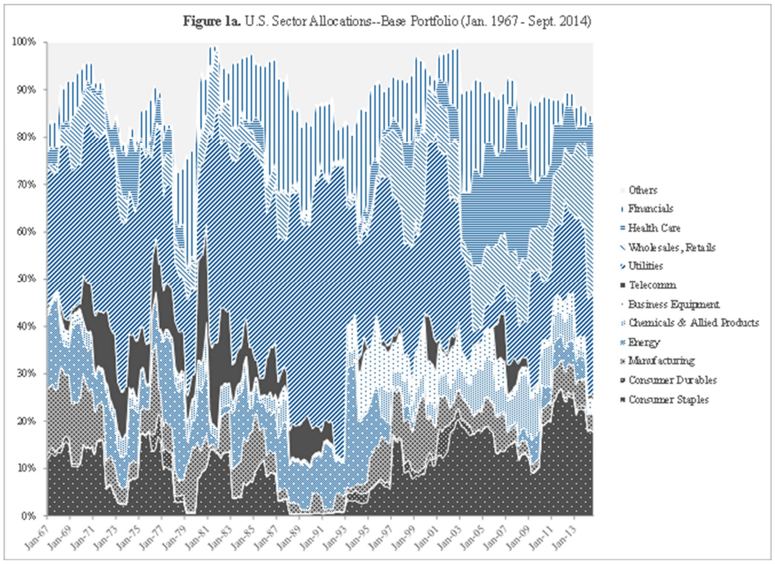
Figure 1a. U.S. Sector Allocations--Base Portfolio (Jan. 1967 - Sept. 2014)
Others
Financials
Health Care
Wholesales, Retails
Utilities
Telecomm
Business Equipment
Chemicals & Allied Products
Energy
Manufacturing
Consumer Durables
Consumer Staples

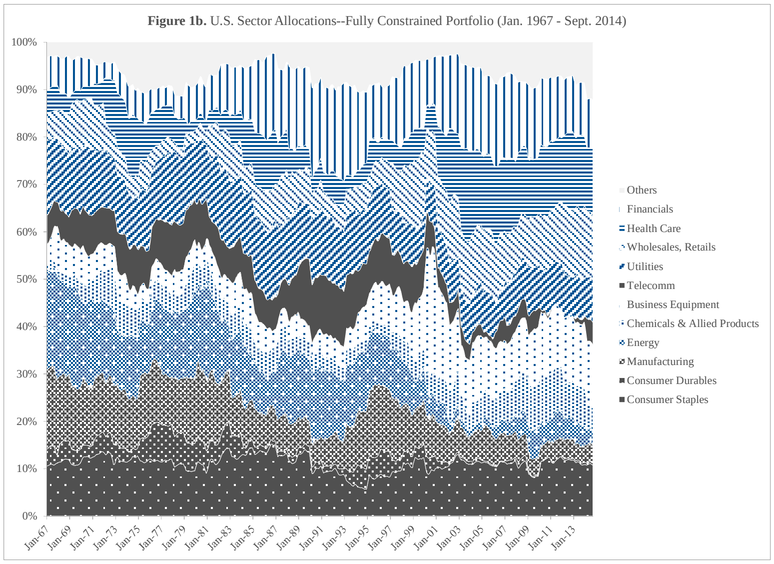
Figure 1b. U.S. Sector Allocations--Fully Constrained Portfolio (Jan. 1967 - Sept. 2014)
Others
Financials
Health Care
Wholesales, Retails
Utilities
Telecomm
Business Equipment
Chemicals & Allied Products
Energy
Manufacturing
Consumer Durables
Consumer Staples

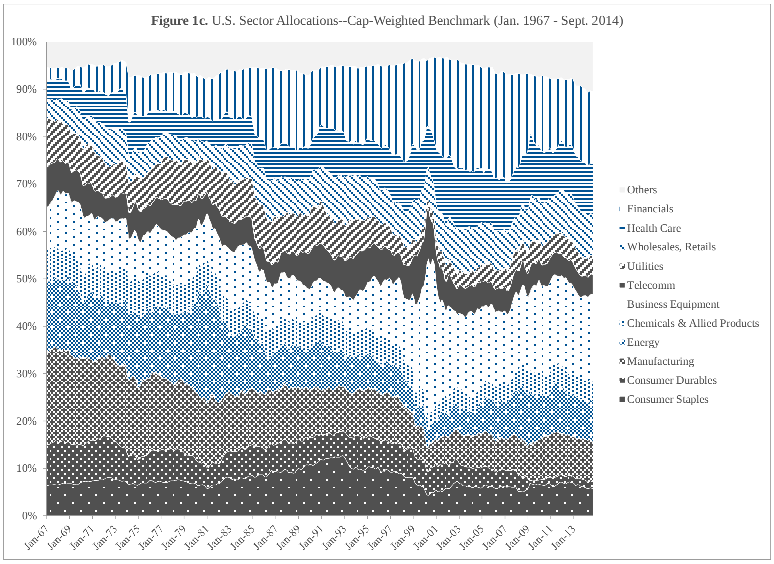
Figure 1c. U.S. Sector Allocations--Cap-Weighted Benchmark (Jan. 1967 - Sept. 2014)
Others
Financials
Health Care
Wholesales, Retails
Utilities
Telecomm
Business Equipment
Chemicals & Allied Products
Energy
Manufacturing
Consumer Durables
Consumer Staples

Figure 2a. Developed World Regional Allocations--Base Portfolio (Jan. 1987 - Sept. 2014)

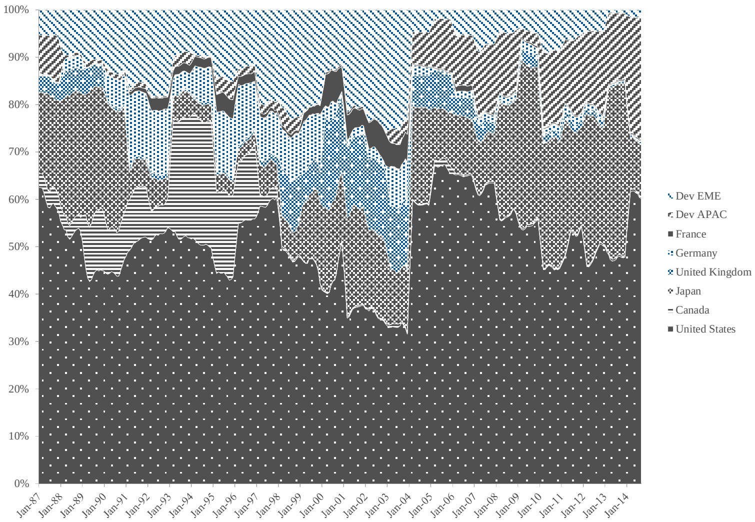

Figure 2b. Developed World Regional Allocations--Fully Constrained Portfolio (Jan. 1987 - Sept. 2014)

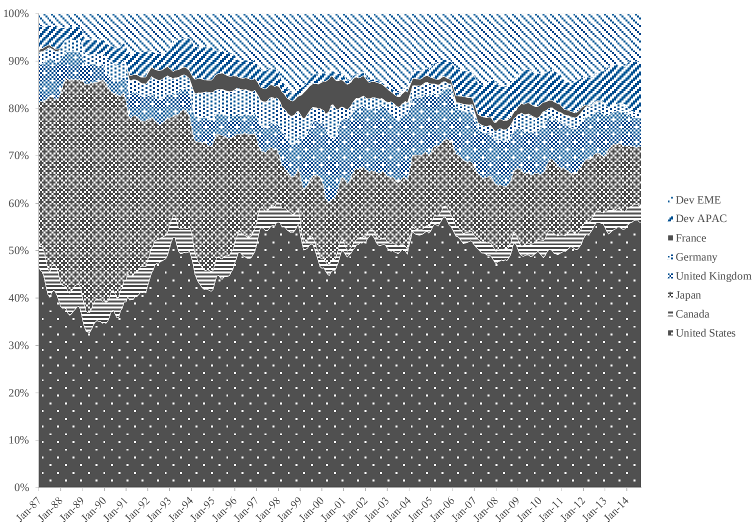

Figure 2c. Developed World Regional Allocations--Cap-Weighted Benchmark (Jan. 1987 - Sept. 2014)

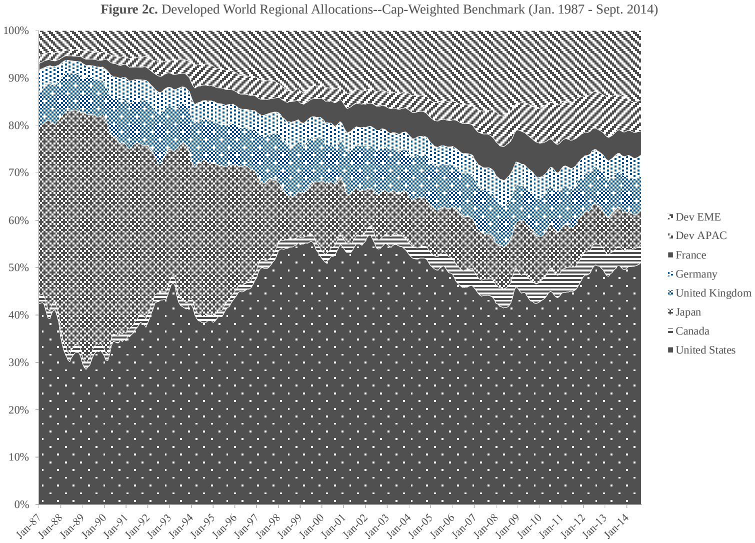

Figure 3a. Emerging World Regional Allocations--Base Portfolio (Jan. 2002 - Sept. 2014)

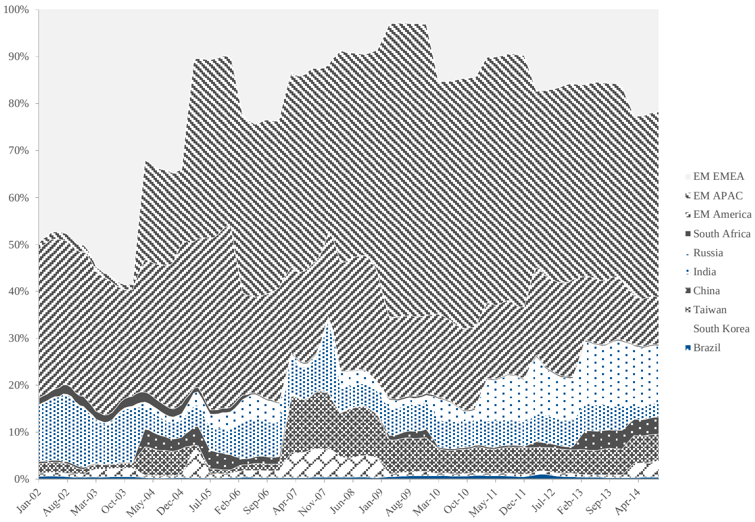

Figure 3b. Emerging World Regional Allocations--Base Portfolio (Jan. 2002 - Sept. 2014)

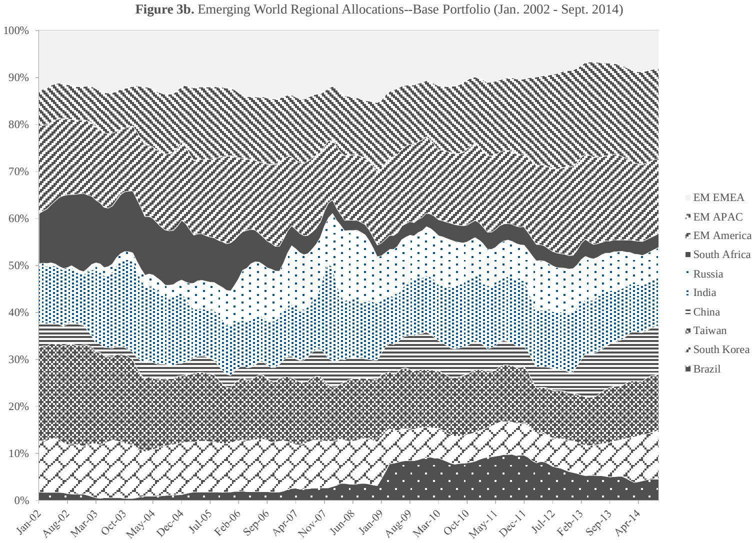

Figure 3c. Emerging World Regional Allocations--Cap-Weighted Benchmark (Jan. 2002 - Sept. 2014)

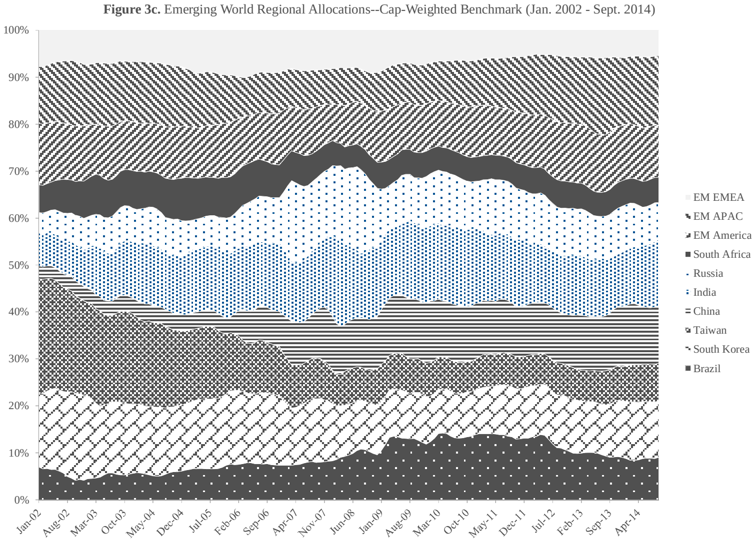

Table 3. Risk Factor Attributions

Panel A: United States (Jan. 1967 - June 2014)	Alpha	Alpha (t-stat)	Market	SMB	HML	WML	BAB	R^2
Simulated Cap-Weighted Benchmark	0.00%	.	1	0	0	0	0	100.00%
Base	-0.30%	-0.32	0.68*	0.13*	0.17*	0	0.29*	77.70%
Add Capacity Constraint	-0.60%	-0.98	0.75*	0.03	0.09*	-0.03*	0.30*	87.90%
Add Sector Concentration Constraint	-0.20%	-0.3	0.81*	0.04*	0.05*	-0.01	0.26*	92.30%
Add Turnover Constraint	0.00%	-0.06	0.82*	0.04*	0.06*	-0.02	0.22*	93.10%
Panel B: Developed Markets (Nov. 1990 - June 2014)	**Alpha**	**Alpha (t-stat)**	**Market**	**SMB**	**HML**	**WML**	**BAB**	R^2
Simulated Cap-Weighted Benchmark	0.00%	.	1	0	0	0	0	100.00%
Base	-1.00%	-0.77	0.53*	-0.05	0.03	-0.05	0.42*	62.00%
Add Capacity Constraint	-0.20%	-0.15	0.59*	-0.07	-0.02	-0.04	0.40*	72.30%
Add Region Concentration Constraint	-1.00%	-0.93	0.64*	-0.08	-0.03	-0.07*	0.44*	79.30%
Add Sector Concentration Constraint	-0.90%	-0.99	0.71*	-0.09*	-0.03	-0.04	0.35*	84.30%
Add Turnover Constraint	-0.30%	-0.34	0.74*	-0.05	-0.03	-0.07*	0.32*	88.80%
Panel C: Emerging Markets (Jan. 2002 - June 2014)	**Alpha**	**Alpha (t-stat)**	**Market**	**SMB**	**HML**	**WML**	**BAB**	R^2
Simulated Cap-Weighted Benchmark	0.00%	.	1	0	0	0	0	100.00%
Base	6.6%*	2.56	0.41*	-0.07	-0.06	0.06	0.23	61.40%
Add Capacity Constraint	5.3%*	2.38	0.56*	-0.11	-0.02	0.10*	0.32*	80.70%
Add Region Concentration Constraint	3.30%	1.83	0.62*	-0.09	-0.03	0.10*	0.33*	88.60%
Add Sector Concentration Constraint	2.10%	1.25	0.65*	-0.06	-0.06	0.12*	0.32*	90.30%
Add Turnover Constraint	3.10%	1.91	0.69*	-0.01	-0.07	0.11*	0.25*	91.70%

*95% statistical significance

Source: Research Affiliates using data from Compustat, CRSP, Worldscope, and Datastream

Table 4. Impact of Individual Constraints

Panel A: United States (1967 - 2014 Sep)	Return	Volatility	Turnover	WAMC Ratio	Effective *N*	Implicit Cost Proxy *	Cost Proxy Relative to Benchmark
Simulated Cap-Weighted Benchmark	10.3%	15.4%	4.7%	100.0%	150	0.33	1.00
Base	12.0%	12.1%	49.5%	20.5%	34	102.10	307.63
Base with Capacity Constraint	11.2%	12.3%	36.7%	33.1%	89	12.95	39.02
Base with Sector Concentration Constraint	11.7%	12.5%	51.6%	35.4%	36	51.91	156.40
Base with Turnover Constraint	11.2%	12.0%	20.5%	24.7%	48	21.79	65.64
Panel B: Developed Markets (1987 - 2014 Se	**Return**	**Volatility**	**Turnover**	**WAMC Ratio**	**Effective *N***	**Implicit Cost Proxy ***	**Cost Proxy Relative to Benchmark**
Simulated Cap-Weighted Benchmark	7.7%	15.6%	6.5%	100.0%	329	0.21	1.00
Base	7.4%	10.3%	49.7%	27.1%	42	52.87	246.62
Base with Capacity Constraint	8.5%	10.9%	40.4%	38.0%	92	12.80	59.73
Base with Region Concentration Constraint	7.5%	10.9%	50.9%	29.1%	42	49.81	232.38
Base with Sector Concentration Constraint	8.0%	11.0%	51.4%	31.3%	43	45.26	211.13
Base with Turnover Constraint	7.8%	11.3%	20.4%	32.1%	55	14.21	66.31
Panel C: Emerging Markets (2002 - 2014 Se	**Return**	**Volatility**	**Turnover**	**WAMC Ratio**	**Effective *N***	**Implicit Cost Proxy ***	**Cost Proxy Relative to Benchmark**
Simulated Cap-Weighted Benchmark	13.2%	22.2%	8.4%	100.0%	218	0.41	1.00
Base	16.4%	12.1%	43.6%	17.6%	33	82.47	199.24
Base with Capacity Constraint	19.1%	14.5%	36.5%	21.1%	97	19.00	45.90
Base with Region Concentration Constraint	13.8%	13.1%	49.0%	21.5%	35	70.64	170.67
Base with Sector Concentration Constraint	15.2%	12.4%	45.3%	20.7%	34	68.22	164.82
Base with Turnover Constraint	18.2%	12.5%	20.2%	17.3%	39	32.15	77.66

* Implicit Cost Proxy calculated as historical average of [turnover / (WAMC relative to benchmark x Effect N)] x 1000

Source: Research Affiliates using data from Compustat, CRSP, Worldsource, and Datastream

Table 5. Various Levels of Capacity Constraints

Developed Markets (Jan. 1987 - Sept. 2014)	Return	Volatility	Turnover	WAMC Ratio	Effective *N*	Implicit Cost Proxy *	Cost Proxy Relative to Benchmark
Simulated Cap-Weighted Benchmark	7.7%	15.6%	6.5%	100.0%	329	0.21	1.00
Base	7.4%	10.3%	49.7%	27.1%	42	52.87	246.62
Base with Capacity Constraint: max wgt = lower of 20x benchmark	8.4%	10.8%	41.5%	43.8%	69	15.74	73.41
Base with Capacity Constraint: max wgt = lower of 20x benchmark	8.4%	10.9%	40.8%	40.0%	81	14.09	65.72
Base with Capacity Constraint: max wgt = lower of 20x benchmark	8.5%	10.9%	40.4%	38.0%	92	12.80	59.73

* Implicit Cost Proxy calculated as historical average of [turnover / (WAMC relative to benchmark x Effect N)] x 1000

Source: Research Affiliates using data from Compustat, CRSP, Worldscope, and Datastream

Table 6. Various Levels of Turnover Constraints

Developed Markets (Jan. 1987 - 2014 Sep)	Return	Volatility	Turnover	WAMC Ratio	Effective *N*	Implicit Cost Proxy *	Cost Proxy Relative to Benchmark
Simulated Cap-Weighted Benchmark	7.7%	15.6%	6.5%	100.0%	329	0.21	1.00
Base	7.4%	10.3%	49.7%	27.1%	42	52.87	246.62
Base with Turnover Constraint at 50%	7.7%	10.0%	39.9%	30.0%	74	22.78	106.26
Base with Turnover Constraint at 40%	7.7%	10.0%	36.9%	30.5%	71	21.54	100.50
Base with Turnover Constraint at 30%	7.6%	10.2%	30.1%	30.7%	65	19.21	89.60
Base with Turnover Constraint at 20%	7.8%	11.3%	20.4%	32.1%	55	14.21	66.31

*Implicit Cost Proxy calculated as historical average of [turnover / (WAMC relative to benchmark x Effect N)] x 1000

Source: Research Affiliates using data from Compustat, CRSP, Worldscope, and Datastream

Table 7. Developed Market Strategies Constructed with Different Parameters

Panel A: Estimating Covariances with 36 Months of Returns	Return	Volatility	Turnover	WAMC Ratio	Effective *N*	Implicit Cost Proxy *	Cost Proxy Relative to Benchmark
Simulated Cap-Weighted Benchmark	7.7%	15.6%	6.5%	100.0%	329	0.21	1.00
Base	6.9%	10.6%	61.1%	29.6%	48	54.70	255.18
Add Capacity Constraint	7.5%	11.2%	52.7%	39.1%	94	15.62	72.86
Add Region Concentration Constraint	7.4%	11.6%	54.8%	40.4%	95	15.61	72.80
Add Sector Concentration Constraint	7.4%	12.0%	56.0%	42.2%	95	15.51	72.38
Add Turnover Constraint	7.9%	12.1%	20.2%	44.2%	108	4.58	21.38
Panel B: Selecting Constituents from Largest 500 Stocks	**Return**	**Volatility**	**Turnover**	**WAMC Ratio**	**Effective *N***	**Implicit Cost Proxy ***	**Cost Proxy Relative to Benchmark**
Simulated Cap-Weighted Benchmark	7.7%	15.6%	6.5%	100.0%	329	0.21	1.00
Base	7.1%	11.1%	46.8%	48.4%	33	34.43	160.59
Add Capacity Constraint	8.1%	11.5%	40.8%	55.9%	73	11.20	52.23
Add Region Concentration Constraint	8.2%	12.2%	43.5%	56.5%	73	12.00	55.96
Add Sector Concentration Constraint	8.1%	12.7%	46.0%	59.0%	74	11.99	55.93
Add Turnover Constraint	8.5%	12.6%	33.3%	61.2%	88	6.73	31.41
Panel C: Rebalancing Semi-Annually	**Return**	**Volatility**	**Turnover**	**WAMC Ratio**	**Effective *N***	**Implicit Cost Proxy ***	**Cost Proxy Relative to Benchmark**
Simulated Cap-Weighted Benchmark	7.7%	15.6%	6.5%	100.0%	329	0.21	1.00
Base	7.7%	10.1%	72.1%	27.4%	42	75.47	352.08
Add Capacity Constraint	8.3%	10.9%	56.8%	37.6%	92	17.97	83.83
Add Region Concentration Constraint	8.5%	11.3%	60.8%	38.4%	93	19.16	89.40
Add Sector Concentration Constraint	8.4%	11.8%	64.7%	42.0%	93	18.21	84.96
Add Turnover Constraint	8.2%	12.3%	20.5%	44.7%	110	4.43	20.66

*Implicit Cost Proxy is calculated as historical average of [turnover / (WAMC relative to benchmark x Effect N)] x 1000

Source: Research Affiliates using data from Compustat, CRSP, Worldscope, and Datastream

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[1] Soe (2012); Blitz and van Vliet (2012).

[2] Behr, Guttler, and Miebs (2008).

[3] Jagannathan and Ma (2003); Kempf and Memmel (2003); AGIC (2012).

[4] Chow, Hsu, Kuo, and Li (2014).

[5] De Carvalho, Lu, and Moulin (2011).
[6] Jagannathan and Ma (2003).
[7] AGIC (2012) argues that the MSCI Minimum Volatility strategy fails to fully benefit from the low volatility premium because it is over-constrained, and presents evidence that the maximum holding and turnover constraints introduce estimation errors. We observe similar performance effects but focus on how index providers can utilize these and other constraints effectively to improve investability.
[8] See Table 2. In developed markets, over the period from January 1987 to September 2014, the simulated Sharpe ratio of the baseline portfolio was 0.38; that of the fully constrained portfolio was modestly higher at 0.40.
[9] See Table 1.
[10] See Appendix for regional composition.
[11] Available methods include the Sharpe (1964) factor-based approach, the Elton and Gruber (1973) constant correlation approach, and the Ledoit and Wolf (2004) statistical shrinkage approach. To estimate the covariance matrix, MSCI Minimum Volatility strategies utilize the Barra Equity Model (MSCI 2013). FTSE Minimum Variance strategies employ the PCA method (FTSE 2014).
[12] Chow, Hsu, Kuo, and Li (2014).
[13] Ledoit and Wolf (2004).
[14] Bengtssson and Holst (2002); Fujiwara et al. (2006).
[15] Jagannathan and Ma (2003); Briner and Connor (2008).
[16] Bayesian shrinkage assumes a target covariance value and shrinks outlying covariance estimates toward the target; other structured factor models assume that certain risk drivers such as the size and value factors explain most of the co-movements in stock returns.
[17] Kwan 2010.
[18] Jagannathan and Ma (2003); Roncalli (2011).
[19] According to the 12-Industry definition from French's data library (http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html).
[20] Leading index providers generally employ a similar approach. When the optimizer fails to converge, it tries again with constraints relaxed in predetermined order and step sizes.
[21] Keim and Madhavan (1997), Huberman and Stanzl (2004), Almgren et al. (2005) and Gatheral (2008).
[22] $Effective\ N = \ (\sum_i w_i{}^2)^{-1}$. See Bouchaud, Potters and Marc (1997).
[23] Hypothetically a portfolio of 100% weight in 1 stock has an effective *N* of 1; a portfolio of equal weight to 1,000 stocks has an effective *N* of 1,000. In another words, these minimum variance portfolios are as diversified as equally weighting only 30-40 stocks.
[24] Another way to visualize concentration is to inspect the top holdings, which in aggregate shed light on how aggressively the optimizer targets the optimal solution. We observe the top 10 holdings of the base strategy and the constrained strategy, which have maximum allowable weights of 5% and 1.5%, often accumulate to 50% and 15%.
[25] This relationship is derived from Aked and Moroz (2015) with the additional assumption that the weights of the minimum variance strategy are independent of the stocks' trading volume. This assumption is consistent with the fact that the minimum variance optimizer takes only the covariance matrix as input.
[26] The minimum variance strategy in Chow, Hsu, Kuo, and Li (2014) is similar to the base strategy. We confirm their findings that the attractive minimum variance strategy performance comes with implementation issues.
[27] The five strategies are the base minimum variance portfolio; the base portfolio with the capacity constraint; the base portfolio with the capacity and sectoral constraints; the base portfolio with the capacity, sectoral, and regional constraints; and the base portfolio with the capacity, sectoral, regional, and turnover constraints.
[28] Hsu, Kalesnik and Li (2012) and Li (2013).
[29] We obtained U.S. and Global factor returns from Professor Kenneth French's website, http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html. We follow the methodology set forth in Fama and French (2012) to simulate size, value and momentum factor returns in the emerging markets.
[30] We follow the methodology described in Frazzini and Pedersen (2014) to simulate BAB factor returns, with slight modifications to mitigate the impact of outliers: We exclude stocks in the bottom 2% of cumulative market capitalization, and those in the top and bottom 1% in beta, from the BAB factor portfolios.
[31] Since BAB is highly correlated with HML in our sample periods, including BAB in the factor regression saps the significance of HML loadings. The regression results excluding BAB are available from the authors upon request.
[32] In multiple years, the turnover occasioned by selling stocks that dropped off the list of the largest 500 stocks is above 15%, leaving very little turnover budget for the optimizer to come up with a solution satisfying all other constraints. Active managers or sophisticated index providers may use complicated rules such as augmenting the

starting universe with existing holdings even if they have dropped out of the large-cap space. For simplicity, we allow higher turnover in our study.

[33] See MSCI (2013) and FTSE (2014).